Construction Algorithms for (Polynomial) Lattice Points

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Joint work with

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Introduction and Motivation

Consider integration of functions on $[0,1]^d$,

$$I_d(f) = \int_{[0,1]^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,,$$

where $f \in \mathcal{H}$, and $\mathcal{H}$ is some Banach space.

Approximate I_d by a quasi-Monte Carlo (QMC) rule,

$$I_d(f) \approx Q_{N,d}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(\boldsymbol{x}_k),$$

where $\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$.

Both parameters d and N can be (very) large.

Worst case error in Banach space $\mathcal{H}$ with respect to $\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$:

$$e_{N,d}(\mathcal{H}, \mathcal{P}_N) := \sup_{f \in \mathcal{H}, \|f\| \le 1} |I_d(f) - Q_{N,d}(f)| .$$

Need $\mathcal{P}_N$ that makes $e_{N,d}(\mathcal{H}, \mathcal{P}_N)$ small: How can we find it?

Function space considered here:

Weighted Korobov space $(\mathcal{H}_{d,\alpha,\gamma}, \|\cdot\|_{d,\alpha,\gamma})$:

space of 1-periodic continuous functions f, where

$$\|f\|_{d,\alpha,\gamma}^2 = \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \rho_{\alpha,\gamma}(\boldsymbol{h}) \, |\widehat{f}(\boldsymbol{h})|^2,$$

where $\widehat{f}(\boldsymbol{h})$is the $\boldsymbol{h}$-th Fourier coefficient of f.

The function $\rho_{\alpha,\gamma}(\boldsymbol{h})$ moderates the decay of the Fourier coefficients.

Set $\boldsymbol{\rho}_{\alpha,\boldsymbol{\gamma}}(\mathbf{0}) := 1$.

For $\boldsymbol{h} = (h_1, \ldots, h_d)$, let $\mathfrak{u} \subseteq \{1, \ldots, d\}$ be the set of the j with $h_j \neq 0$. Then

$$\boldsymbol{\rho}_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = \gamma_{\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} |h_j|^{\alpha} .$$

- $\alpha > 1$: "smoothness parameter" (higher $\alpha \to$ smoother functions in $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}}$),
- $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,d\}}$: coordinate weights.

Weights?

- Sloan and Woźniakowski (1998): Assign weights to different groups of coordinates to model their different influence on a problem:

$$\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,d\}}$$

of positive reals: weights.

- larger weights $\simeq$ more influence of corresponding variables, smaller weights $\simeq$ less influence of corresponding variables.
- Suitable weights can help to reduce negative influence of the dimension.
- Important class of weights: product weights

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$$

for positive reals γ_j. In this case, assume

$$1 = \gamma_1, \quad \text{and} \quad \gamma_j \geq \gamma_{j+1} \quad \text{for all } j.$$

Here:

$\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a rank-1 lattice point set with generating vector

$$\boldsymbol{z} = (z_1, \ldots, z_d) \in \{1, \ldots, N-1\}^d.$$

Points of $\mathcal{P}_N$:

$$\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,d})$$

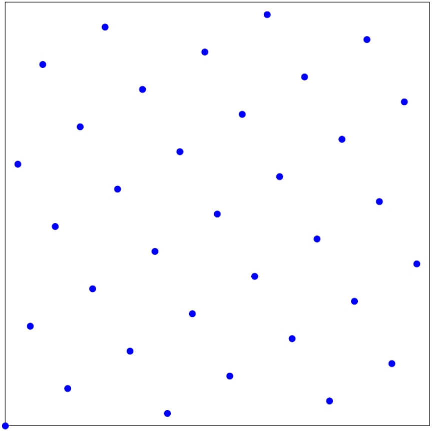
with

$$x_{n,j} = \left\{ \frac{n z_j}{N} \right\},$$

where $\{t\} = t - \lfloor t \rfloor$.

Note: Given N and d, $\boldsymbol{z}$ fully determines the lattice point set.

Lattice point set with $d = 2$, $N = 34$, and $\boldsymbol{z} = (1, 21)$:



Worst-case error of a lattice rule with generating vector $\boldsymbol{z}$:

$$e^2_{N,d,\alpha,\gamma}(\boldsymbol{z}) = \sum_{\substack{\boldsymbol{0}\neq\boldsymbol{h}\in\mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \bmod N}} (\rho_{\alpha,\gamma}(\boldsymbol{h}))^{-1}.$$

Explicit formula for the (squared) worst-case error for product weights:

$$e^2_{N,d,\alpha,\gamma}(\boldsymbol{z}) = -1 + \frac{1}{N}\sum_{n=0}^{N-1}\prod_{j=1}^{d}\left(1+\gamma_j\varphi_\alpha\left(\left\{\frac{nz_j}{N}\right\}\right)\right),$$

where $\varphi_\alpha\left(\frac{k}{N}\right)$ can be computed for all values of $k = 0,\ldots,N-1$.

The error formula is easy to implement.

Question: is the Korobov space interesting?

- Yes, first reason: it is a model function space that makes it easier to understand how lattice rules work.
- Yes, second reason:

  Bounds on the worst-case error of lattice rules in the Korobov space with $\alpha = 2$ immediately yield bounds on the worst-case error of slightly modified lattice rules in certain unanchored Sobolev spaces.

- All that remains is to find “good” $\boldsymbol{z} \in \{1, \ldots, N-1\}^d$.
- Huge search space of size $(N-1)^d$. (e.g., $N = 10\,000$ and $d = 100$).
- Component by component (CBC) construction: greedy algorithm to construct z_j one at a time.
  Size of search space is $N-1$ per component.
  (Almost) optimal convergence of error bounds (Kuo, 2003).

Algorithm 1 (CBC construction)

Let N be given. Construct $\boldsymbol{z} = (z_1, \ldots, z_d)$ as follows.

- *Set $z_1 = 1$.*
- *For $s \in \{2, \ldots, d\}$ assume that $z_1, \ldots, z_{s-1}$ have already been found. Now choose $z_s \in \{1, \ldots, N-1\}$ such that*

$$e^2_{N,s,\alpha,\gamma}((z_1, \ldots, z_{s-1}, z_s))$$

  is minimized as a function of z_s.
- *Increase s and repeat the second step until $(z_1, \ldots, z_d)$ is found.*

- Can do fast CBC (Cools, Nuyens, 2006): computation cost of $\mathcal{O}(dN \log N)$.
- Computation cost of $\mathcal{O}(dN \log N)$ can still be demanding for big N, d.
- Might want to have big N, d simultaneously. $\rightarrow$ **Can we speed up the search?**

The reduced CBC construction
(joint work with J. Dick, G. Leobacher, F. Pillichshammer)

Idea: small weights for later components $\rightarrow$ make search space smaller for later components z_j.

- Let N be a prime power, $N = b^m$, b prime, $m \in \mathbb{N}$.
- Let $w_1, w_2, \ldots \in \mathbb{N}_0$ with $0 = w_1 \le w_2 \le \cdots$.
- Consider the sequence of reduced search spaces

$$\mathcal{Z}_{N,w_j} := \begin{cases} \{1 \le z < b^{m-w_j} : \gcd(z,N) = 1\} & \text{if} \quad w_j < m, \\ \{1\} & \text{if} \quad w_j \ge m. \end{cases}$$

- Note that

$$|\mathcal{Z}_{N,w_j}| := \begin{cases} b^{m-w_j-1}(b-1) & \text{if} \quad w_j < m, \\ 1 & \text{if} \quad w_j \ge m. \end{cases}$$

- write $Y_j := b^{w_j}$.

Algorithm 2 (Reduced CBC construction)

Let N, $w_1, \ldots, w_d$, and $Y_1, \ldots, Y_d$ be as above. Construct $\boldsymbol{z} = (Y_1 z_1, \ldots, Y_d z_d)$ as follows.

- *Set $z_1 = 1$.*
- *For $s \in \{2, \ldots, d\}$ assume that $z_1, \ldots, z_{s-1}$ have already been found. Now choose $z_s \in \mathcal{Z}_{N,w_s}$ such that*

$$e^2_{N,s,\alpha,\gamma}((Y_1 z_1, \ldots, Y_{s-1} z_{s-1}, Y_s z_s))$$

  is minimized as a function of z_s.
- *Increase s and repeat the second step until $(Y_1 z_1, \ldots, Y_d z_d)$ is found.*

Usual CBC construction: $w_j = 0$ and $Y_j = 1$ for all j.

Theorem 3 (Dick/K./Leobacher/Pillichshammer, 2015)

Let $\boldsymbol{z} = (Y_1 z_1, \ldots, Y_d z_d) \in \mathbb{Z}^d$ *be constructed according to the reduced CBC algorithm. Then,*

$$e_{N,d,\alpha,\gamma}((Y_1 z_1, \ldots, Y_d z_d)) \leq$$
$$\leq N^{-\alpha/2+\delta} \left(2 \prod_{j=1}^{d} \left(1 + \gamma_j^{\frac{1}{\alpha-2\delta}} 2\zeta\left(\frac{\alpha}{\alpha-2\delta}\right) b^{w_j} \right) \right)^{\alpha/2-\delta}$$

for all $\delta \in \left(0, \frac{\alpha-1}{2}\right]$*, where* ζ *is the Riemann zeta function.*

Theorem formulated for product weights, similar result holds for general weights.

If

$$B := \sum_{j=1}^{\infty} \gamma_j^{\frac{1}{\alpha - 2\delta}} b^{w_j} < \infty,$$

then the error bound is independent of the dimension.

- Fast CBC construction (Nuyens/Cools) for non-reduced case ($w_j = 0$) has computation cost of $\mathcal{O}(dN \log N)$.
- Idea also works for the reduced case (assuming product weights), yields reduced cost by exploiting additional structure of the case $w_j > 0$.
- Bonus: once $w_j \geq m$ the search space contains only one element, construction of additional components incurs no extra cost.
- Computational cost of the reduced fast CBC construction is

$$\mathcal{O}\left(N \log N + \min\{d, d^*\} N + \sum_{j=1}^{\min\{d,d^*\}} (m - w_j) N b^{-w_j} \right),$$

where $d^* := \max\{j \in \mathbb{N} : w_j < m\}$.

Computation times (in seconds) for $b = 2$, $\alpha = 2$, $\gamma_j = 0.7^j$:

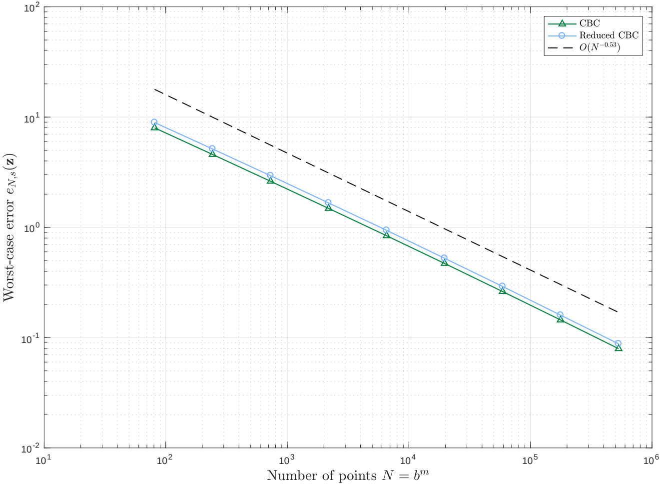
$w_j = 0$ (non-reduced)

$w_j = \lfloor 3 \log_b j \rfloor$ (reduced)

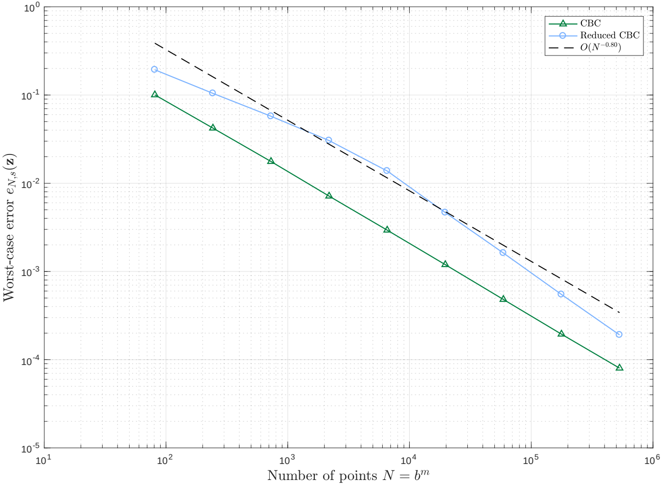
	$d = 100$	$d = 500$	$d = 1000$	d^*
$m = 10$	0.0329 0.0021	0.1600 0.0022	0.3230 0.0022	10
$m = 14$	0.0851 0.0076	0.4380 0.0105	0.8560 0.0071	25
$m = 18$	0.8320 0.0897	4.4400 0.0915	8.5400 0.0910	63
$m = 20$	4.1700 0.6230	21.5000 0.6430	42.4000 0.6360	101

fast CBC (top), reduced fast CBC (bottom).

Error convergence:
$b = 3$, $\alpha = 2$, $d = 100$, $\gamma_j = 0.8^j$:



Error convergence:
$b = 3$, $\alpha = 2$, $d = 100$, $\gamma_j = j^{-3}$:



The reduced fast CBC construction for POD weights (joint work with A. Ebert, D. Nuyens)

Recent result:

- Adaption of the reduced fast CBC construction to POD weights (motivated by a question by Ch. Schwab).
- POD weights: product and order-dependent weights, relevant for applications in PDEs with random coefficients:

$$\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|} \prod_{j \in \mathfrak{u}} \gamma_j .$$

- Reduced fast CBC construction for this class of weights.
- Yields a slight improvement for the special case of product weights.

Theorem 4 (Ebert/K./Nuyens, 2019)

Given a sequence $0 \leq w_1 \leq w_2 \leq \cdots$ *and POD weights we can construct a lattice rule with* $N = b^m$ *points in* d *dimensions, whose worst-case error has almost optimal convergence, with construction cost of order*

$$\mathcal{O}\left(\sum_{j=1}^{\min\{d,d^*\}} (m - w_j + j)\, b^{m-w_j}\right),$$

The memory cost is $\mathcal{O}(\sum_{j=1}^{\min\{d,d^*\}} b^{m-w_j})$.

In case of product weights we need $\mathcal{O}(\sum_{j=1}^{\min\{d,d^*\}} (m - w_j)\, b^{m-w_j})$ *for the construction with memory* $\mathcal{O}(b^{m-w_1})$.

The successive coordinate search (SCS) construction

Question: can one improve on the quality of the output vector of the CBC construction?

Ebert/Leövey/Nuyens (2016): successive coordinate search (SCS) construction.

Basic idea:

- Assume product weights.
- Begin with a start vector $\boldsymbol{z}^{(0)} = (z_1^{(0)}, \ldots, z_d^{(0)})$.
- Update components one after the other by minimizing worst-case error.
- Returned vector is at least as good as initial vector.

Algorithm 5 (SCS construction)

Let N be a prime. Let $\boldsymbol{z}^{(0)} = (z_1^{(0)}, \ldots, z_d^{(0)}) \in \{0, 1, \ldots, N-1\}^d$ be given.

- *For $s \in \{1, \ldots, d\}$ assume that $z_1, \ldots, z_{s-1}$ have already been found. Now choose $z_s \in \{1, \ldots, N-1\}$ such that*

$$e^2_{N,d,\alpha,\gamma}((z_1, \ldots, z_{s-1}, z_s, z_{s+1}^{(0)}, \ldots, z_d^{(0)}))$$

  is minimized as a function of z_s.
- *Increase s and repeat the second step until $\boldsymbol{z} = (z_1, \ldots, z_d)$ is found.*

Theorem 6 (Ebert/Leövey/Nuyens, 2016)

Let $\boldsymbol{z}^{(1)}$ be constructed by the fast CBC algorithm.
Set $\boldsymbol{z}^{(0)} := \boldsymbol{z}^{(1)}$ in the SCS algorithm.
Let $\boldsymbol{z}$ be the vector returned by SCS. Then

$$e^2_{N,d,\alpha,\gamma}(\boldsymbol{z}) \le e^2_{N,d,\alpha,\gamma}(\boldsymbol{z}^{(1)}).$$

Theorem 7 (Ebert/Leövey/Nuyens, 2016)

Set $\boldsymbol{z}^{(0)} := (0, 0, \ldots, 0)$ in the SCS construction.
Then the SCS construction yields the same result as usual CBC construction. SCS is thus a generalization of CBC.

- SCS yields improvements over the CBC construction for pre-asymptotically moderately decreasing weights, e.g.

$$\gamma_j = 0.95^j.$$

- There is a fast implementation of the SCS construction:
  - Pre-computation with cost of $\mathcal{O}(dN)$,
  - matrix-vector multiplication of same speed as in CBC,
  - total cost of $\mathcal{O}(dN \log N)$.

The reduced fast SCS construction (joint work with A. Ebert)

Combine advantages of reduced fast CBC construction and fast SCS construction:
reduced fast SCS construction.

Same setting as for reduced fast CBC construction, brings similar results:

- Reduction in computation times if weights decay fast,
- similar error convergence behavior.
- Modify SCS construction to prime-power N,
- Generalization to arbitrary (not only product) weights.

The CBC-DBD construction
(ongoing joint work with A. Ebert, D. Nuyens, O.O. Osisiogu)

Slightly different function class (based on Korobov's original work):

$$E_{d,\gamma}^{\alpha} := \Big\{ f \in L^2([0,1]^d) \mid \exists\, C > 0 \text{ such that } |\hat{f}(\boldsymbol{h})| \le C\,(\rho_{\alpha,\gamma}(\boldsymbol{h}))^{-1}\ \forall\, \boldsymbol{h} \in \mathbb{Z}^d \Big\}.$$

$E_{d,\gamma}^{\alpha}$ is a Banach space with norm given by

$$\|f\|_{E_{d,\gamma}^{\alpha}} := \sup_{\boldsymbol{h} \in \mathbb{Z}^d} |\hat{f}(\boldsymbol{h})|\, \rho_{\alpha,\gamma}(\boldsymbol{h}).$$

Worst-case error of a lattice rule with generating vector $\boldsymbol{z}$:

$$\widetilde{e}_{N,d,\alpha,\gamma}(\boldsymbol{z}) = \sum_{\substack{\boldsymbol{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \bmod N}} (\rho_{\alpha,\gamma}(\boldsymbol{h}))^{-1}.$$

This yields a connection between $E_{d,\gamma}^{\alpha}$ and $(\mathcal{H}_{d,\alpha,\gamma}, \|\cdot\|_{d,\alpha,\gamma})$:

$$\widetilde{e}_{N,d,\alpha,\gamma}(\boldsymbol{z}) = e_{N,d,\alpha,\gamma}^{2}(\boldsymbol{z}).$$

Results on

$$\widetilde{e}_{N,d,\alpha,\gamma}(\boldsymbol{z}) \quad (\text{for } E_{d,\gamma}^{\alpha})$$

thus automatically imply results on

$$e_{N,d,\alpha,\gamma}^{2}(\boldsymbol{z}) \quad (\text{for } \mathcal{H}_{d,\alpha,\gamma}, \|\cdot\|_{d,\alpha,\gamma}).$$

Recall our goal: Efficiently construct a “good” generating vector

$$\boldsymbol{z} = (z_1, \ldots, z_d) \in \{0, \ldots, N-1\}^d$$

of a rank-1 lattice rule.

Assume $N = 2^n$ for some $n \in \mathbb{N}$.

For $s \in \{1, \ldots, d\}$ we write

$$z_s = z_{s,1} + z_{s,2}\, 2 + \cdots + z_{s,n}\, 2^{n-1}$$

with $z_{s,v} \in \{0, 1\}$.

- **Idea by Korobov:** Construct the digits $z_{s,v} \in \{0, 1\}$, $1 \leq v \leq n$, one after the other.
- This yields a digit-by-digit (DBD) construction.
- Go through the DBD-construction component-wise.
- This gives a CBC-DBD construction.
- Construction is extensible in d, but not in the number of digits (points).

To this end: Define a complicated quality function:

Let $x \in \mathbb{N}$ be odd, and $n, s \in \mathbb{N}$, and let $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,d\}}$ be positive weights. For $1 \le v \le n$ and $1 \le s \le d$ let

$$h_{s,v,\gamma}(x) := \sum_{k=v}^{n} \frac{1}{2^{k-v}} \sum_{\substack{m=1 \\ m \equiv 1 \bmod 2}}^{2^k} \left(\sum_{\emptyset \ne \mathfrak{u} \subseteq \{1,\ldots,s-1\}} \gamma_{\mathfrak{u}} \prod_{j \in \mathfrak{u}} \ln \frac{1}{\sin^2(\pi m a_{j,n}/2^k)} \right.$$
$$\left. + \sum_{\mathfrak{w} \subseteq \{1,\ldots,j-1\}} \gamma_{\mathfrak{w} \cup \{s\}} \left(\prod_{j \in \mathfrak{w}} \ln \frac{1}{\sin^2(\pi m a_{j,n}/2^k)} \right) \ln \frac{1}{\sin^2(\pi m x/2^v)} \right).$$

Use $h_{s,v,\gamma}$ in the following CBC-DBD construction.

Algorithm 8 (CBC-DBD construction)

Given $N = 2^n$, $d \in \mathbb{N}$, *and* $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\dots,d\}}$.

- *Set* $z_1 = 1$ *and* $a_{2,1} = \cdots = a_{d,1} = 1$.
- *Outer loop (CBC): for* $s \in \{2, \dots, d\}$:
- *Inner loop (DBD): for* $v \in \{2, \dots, n\}$:
- *Choose* $z_{s,v} = \operatorname{argmin}_{z \in \{0,1\}} h_{s,v,\gamma}(a_{s,v-1} + 2^{v-1} z)$.
- *Set* $a_{s,v} = a_{s,v-1} + 2^{v-1} z_{s,v}$.
- *Increase* v; *repeat inner loop until* $v = n$.
- *Increase* s; *repeat outer loop until* $s = d$.

Set $z_s := a_{s,n}$ *for* $2 \leq s \leq d$.

The CBC-DBD construction yields almost optimal results:

Adaption of proof idea by Korobov (1960s):

We write

$$M_d := \{-(N-1), \ldots, N-1\}^d \setminus \{\mathbf{0}\}$$

Note that

$$\begin{aligned} \widetilde{e}_{N,d,\alpha,\gamma}(\boldsymbol{z}) &= \sum_{\substack{\mathbf{0} \neq \boldsymbol{h} \in \mathbb{Z}^d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \bmod N}} \frac{1}{\rho_{\alpha,\gamma}(\boldsymbol{h})} \\ &= \sum_{\substack{\boldsymbol{h} \in M_d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \bmod N}} \frac{1}{\rho_{\alpha,\gamma}(\boldsymbol{h})} + \sum_{\substack{\boldsymbol{h} \notin M_d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \bmod N}} \frac{1}{\rho_{\alpha,\gamma}(\boldsymbol{h})}. \end{aligned}$$

We can easily show:

Proposition 9

Let $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ *with* $\gcd(z_j, N) = 1$ *for all* $j = 1, \ldots, d$. *Then, for* $\alpha > 1$ *we have*

$$\sum_{\substack{\boldsymbol{h} \notin M_d \\ \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \bmod N}} \frac{1}{\rho_{\alpha,\gamma}(\boldsymbol{h})} \leq \frac{1}{N^\alpha} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,d\}} \gamma_{\mathfrak{u}}^{\alpha} (4\zeta(\alpha))^{|\mathfrak{u}|}.$$

The CBC-DBD algorithm constructs a generating vector $\boldsymbol{z} = (z_1, \ldots, z_d)$ such that there exist constants $C(\gamma, \delta)$ with

$$\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,d\}} \gamma_{\mathfrak{u}} \sum_{\substack{\boldsymbol{h}_{\mathfrak{u}} \in M_{|\mathfrak{u}|} \\ \boldsymbol{h}_{\mathfrak{u}} \cdot \boldsymbol{z}_{\mathfrak{u}} \equiv 0 \bmod N}} \frac{1}{\prod_{j \in \mathfrak{u}} |h_j|} \le C(\gamma, \delta) N^{-1+\delta}$$

for any $\delta > 0$, if the weights γ satisfy a suitable summability condition.

This result then yields

Theorem 10 (Ebert/K./Osisiogu/Nuyens, 2019)

Let $N = 2^n$, and let $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1,\ldots,d\}}$ be a sequence of positive weights satisfying

$$\sum_{\substack{\mathfrak{u} \subseteq \mathbb{N} \\ |\mathfrak{u}| < \infty}} |\mathfrak{u}| \, (\gamma_{\mathfrak{u}})^{1/|\mathfrak{u}|} < \infty$$

Let $\boldsymbol{z} = (z_1, \ldots, z_d)$ be the generating vector constructed by the CBC-DBD algorithm. Then for any $\delta > 0$ and $\alpha > 1$ we have

$$\widetilde{e}_{N,d,\gamma^\alpha,\alpha}(\boldsymbol{z}) \leq \frac{1}{N^\alpha} \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,\ldots,d\}} \gamma_u^\alpha (4\zeta(\alpha))^{|\mathfrak{u}|} + C(\gamma, \delta) N^{\alpha\delta} \right)$$

with $\gamma^\alpha = (\gamma_{\mathfrak{u}}^\alpha)_{\mathfrak{u} \subseteq \{1,\ldots,d\}}$ and some positive constant $C(\gamma, \delta)$ independent of d and N.

Note that:

- The proof technique is such that the CBC-DBD algorithm is independent of α.
- Rather elementary methods can be applied to ensure a run-time of $\mathcal{O}(dN \log N)$ of the CBC-DBD algorithm.

Niederreiter & Pillichshammer, 2009:

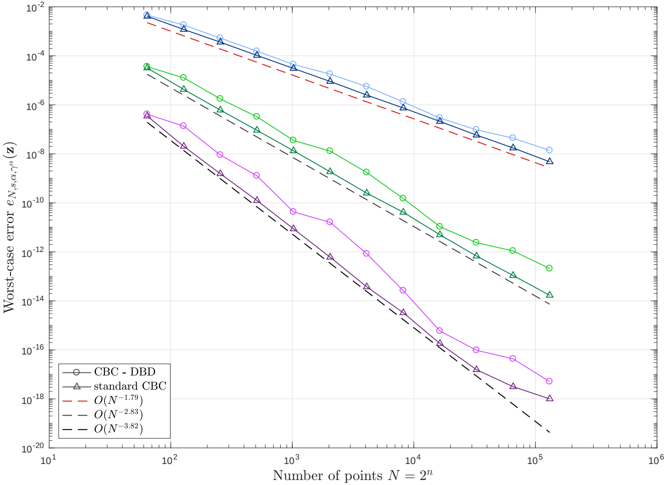
- Idea of DBD construction for extensible (polynomial) lattice rules.
- Rules extensible in the number of points N, but not in dimension d (CBC-DBD construction: extensible in d, but not in N).
- Convergence rate not optimal (CBC-DBD-construction: almost optimal convergence rate possible).

Future research:

Can we obtain better convergence rate for extensible rules using Korobov's original method?

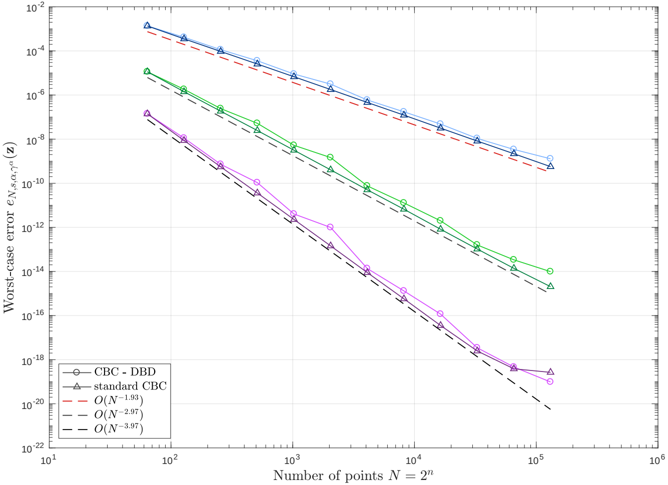
Can we obtain an algorithm extensible in N *and* d?

Error convergence:
$N = 2^n$, $d = 100$, $\gamma_j = j^{-2}$, $\alpha = 2$, $\alpha = 3$, $\alpha = 4$:

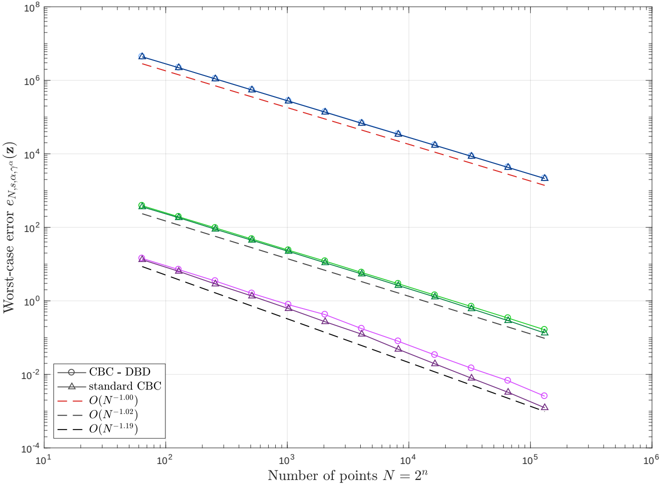


Error convergence:

$N = 2^n$, $d = 100$, $\gamma_j = j^{-3}$, $\alpha = 2, \alpha = 3, \alpha = 4$:

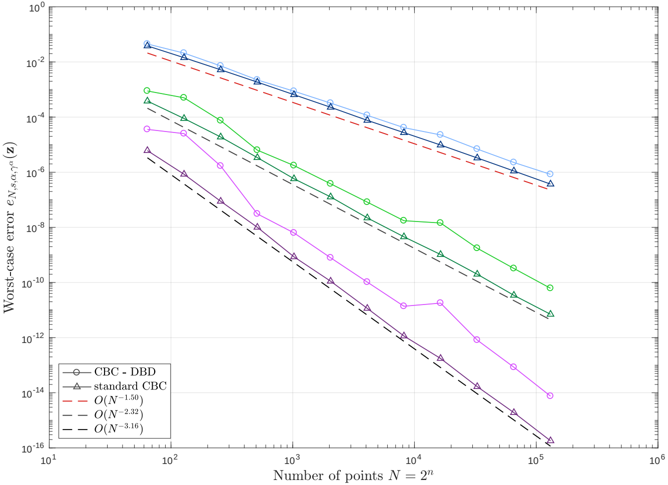


Error convergence:
$N = 2^n$, $d = 100$, $\gamma_j = 0.95^j$, $\alpha = 2$, $\alpha = 3$, $\alpha = 4$:



Error convergence:

$N = 2^n$, $d = 100$, $\gamma_j = 0.7^j$, $\alpha = 2$, $\alpha = 3$, $\alpha = 4$:



Polynomial lattice rules:

- Polynomial lattice rules: similar to lattice rules, but integer arithmetic replaced by polynomial arithmetic over finite fields.
- Special cases of Niederreiter's (t, m, d)-nets, powerful QMC methods.
- CBC, fast CBC, reduced fast CBC, SCS, fast SCS, reduced fast SCS work analogously for polynomial lattice rules.
- CBC-DBD construction for polynomial lattice rules: in progress.

Conclusion

- Component-by-component (CBC) construction of (polynomial) lattice points is a standard method in modern QMC theory.
- Fast CBC construction by Nuyens/Cools needs $\mathcal{O}(dN \log N)$ operations.
- We can reduce the construction cost depending on the coordinate weights, sometimes obtain independence of the dimension.
- The (fast) SCS construction can improve on the quality of generating vectors, as compared to the CBC construction. We can reduce the run-time of the SCS algorithm, similarly to that of the CBC construction.
- CBC-DBD construction seems to be a useful alternative to previous algorithms.

Thanks for your attention.